

# EVALUATION OF THE ADM MASS AND CENTER OF MASS VIA THE RICCI TENSOR

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**ABSTRACT.** We prove directly without using a density theorem that (i) the ADM mass defined in the usual way on an asymptotically flat manifold is equal to the mass defined intrinsically using the Ricci tensor; (ii) the Hamiltonian formulation of center of mass and the center of mass defined intrinsically using the Ricci tensor are the same.

## 1. INTRODUCTION

Let  $(M^n, g)$  be an end of some asymptotically flat manifold, i.e.  $M^n$  is diffeomorphic to  $\mathbb{R}^n \setminus B(1)$ , where  $B(r) = \{|x| < r\}$ , such that in the coordinates  $\{x^i\}$  on  $\mathbb{R}^n$ ,  $g_{ij} - \delta_{ij}$  together with its derivatives decays at infinity, which will be made precise later. We will use the Einstein summation convention throughout this paper, i.e. summation over any pair of repeated indices.

For each large  $r$ , let

$$(1.1) \quad \mathbf{m}(r) = \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,j} - g_{jj,i}) \nu_e^i d\sigma_e,$$

where  $S_r = \partial B(r)$ ,  $\nu_e$  is the unit outward normal and  $d\sigma_e$  is the area element on  $S_r$  with respect to the Euclidean metric and  $\omega_{n-1}$  is the area of the unit sphere in  $\mathbb{R}^n$ . The ADM mass [1] of  $(M^n, g)$  is defined as

$$(1.2) \quad \mathbf{m} = \lim_{r \rightarrow \infty} \mathbf{m}(r)$$

provided the limit exists. Under suitable conditions, it was proved by Bartnik [3] and Chruściel [9] independently that  $\mathbf{m}$  is defined and does not depend on the choice of coordinates.

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In the general relativity literature (see Ashtekar–Hansen [2] and Chruściel [10]), it is also known that the ADM mass  $\mathbf{m}$  can be computed using the curvature of  $g$  as follows<sup>1</sup>: Consider

$$(1.3) \quad \mathbf{m}_I(r) = \frac{1}{(n-1)(2-n)\omega_{n-1}} \int_{S_r} \left( \text{Ric} - \frac{1}{2} R_g g \right) (X, \nu_g) d\sigma_g,$$

where  $\text{Ric}$  and  $R_g$  are the Ricci tensor and the scalar curvature of  $g$  respectively,  $X$  is the Euclidean conformal Killing vector field  $x^i \frac{\partial}{\partial x^i}$ ,  $\nu_g$  is the unit outward normal and  $d\sigma_g$  is the area element on  $S_r$  with respect to  $g$ . If the limit

$$(1.4) \quad \mathbf{m}_I = \lim_{r \rightarrow \infty} \mathbf{m}_I(r)$$

exists, then

$$(1.5) \quad \mathbf{m} = \mathbf{m}_I.$$

Formula (1.5) was also suggested by Schoen (cf. [11, 7]) in connection with the generalized Pohozaev identity [16]. (For a recent application of (1.5) to the content at infinity of asymptotically flat metrics, see [5, Proposition 2.2].) It can be easily checked that (1.5) holds for metrics which are conformally flat (up to higher order) near infinity. Thus a common proof of (1.5) is to apply a density theorem from [8] or [17] to reduce the general case to metrics with harmonic asymptotics. An outline of such an approach was given in [12].

When  $\mathbf{m} \neq 0$ , there exist several notions of center of mass for  $(M, g)$  (cf. [11, 12]). Similar to the definition of  $\mathbf{m}$ , the Hamiltonian formulation of center of mass  $\mathbf{c}_{\text{CS}}$ , proposed by Regge–Teitelboim [14] (also by Beig–Ó Murchadha [4] and Corvino–Schoen [8]) is given as follows: let

$$c_{\text{CS}}^\alpha(r) = \frac{1}{2(n-1)\omega_{n-1}\mathbf{m}} \int_{S_r} [x^\alpha (g_{ij,i} - g_{ii,j}) \nu_e^j - (g_{i\alpha} \nu_e^i - g_{ii} \nu_e^\alpha)] d\sigma_e,$$

where  $\alpha = 1, \dots, n$ . Note that  $g_{i\alpha}, g_{ii}$  in the second term in the integrand can be replaced by  $h_{i\alpha}, h_{ii}$  where  $h_{ij} := g_{ij} - \delta_{ij}$ . Let

$$\mathbf{c}_{\text{CS}}(r) = (c_{\text{CS}}^1(r), c_{\text{CS}}^2(r), \dots, c_{\text{CS}}^n(r)),$$

then

$$(1.6) \quad \mathbf{c}_{\text{CS}} = \lim_{r \rightarrow \infty} \mathbf{c}_{\text{CS}}(r)$$

provided the limit exists. (Here we use the notation  $\mathbf{c}_{\text{CS}}$  following [11].) Similar to (1.4), Schoen suggested an intrinsic way to define the center

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<sup>1</sup>We thank Piotr Chruściel for bringing our attention the references [2] and [10].

of mass (cf. [11]): for  $\alpha = 1, \dots, n$ , let  $Y_{(\alpha)}$  be the Euclidean conformal Killing vector field

$$(|x|^2 \delta^{\alpha i} - 2x^\alpha x^i) \frac{\partial}{\partial x^i},$$

define

$$(1.7) \quad c_I^\alpha(r) = \frac{1}{2(n-1)(n-2)\omega_{n-1}\mathbf{m}} \int_{S_r} \left( \text{Ric} - \frac{1}{2}R_g g \right) (Y_{(\alpha)}, \nu_g) d\sigma_g$$

and

$$\mathbf{c}_I(r) = (c_I^1(r), \dots, c_I^n(r)).$$

The intrinsic center of mass is defined as

$$(1.8) \quad \mathbf{c}_I = \lim_{r \rightarrow \infty} \mathbf{c}_I(r)$$

provided the limit exists. We also want to mention that Huisken-Yau [13] and Ye [18] constructed a foliation of stable constant mean curvature spheres near infinity on asymptotically Schwarzschild manifolds via different methods. Huisken-Yau [13] proposed a geometric definition of center of mass using the foliation. It was proved by Huang [11] that under the Regge-Teitelboim condition (see Theorem 1.1 (b)), all these notions of center of mass are equivalent. In order to show  $\mathbf{c}_{\text{CS}} = \mathbf{c}_I$ , in [11] Huang first proved a density theorem for metrics satisfying the Regge-Teitelboim condition and then apply it to reduce the general case to metrics with harmonic asymptotics. Recently, in [15], Nerz also used  $\mathbf{m}_I$  as the definition of mass in his construction of foliations of constant mean curvature spheres for asymptotically flat manifolds under weaker asymptotic conditions.

In this paper, we give a direct proof of  $\mathbf{c}_{\text{CS}} = \mathbf{c}_I$  and  $\mathbf{m} = \mathbf{m}_I$  without using density theorems. More precise, we will prove the following:

**Theorem 1.1.** *On an asymptotically flat end  $(M^n, g)$  which satisfies  $g_{ij} - \delta_{ij} = o_2(|x|^{-\frac{n-2}{2}})$ , one has*

(a)

$$\lim_{r \rightarrow \infty} (\mathbf{m}(r) - \mathbf{m}_I(r)) = 0.$$

(b) *If the Regge-Teitelboim condition holds, i.e.  $g_{ij}^{\text{odd}}(x) = o_2(|x|^{-\frac{n}{2}})$ , then*

$$\lim_{r \rightarrow \infty} (\mathbf{c}_{\text{CS}}(r) - \mathbf{c}_I(r)) = 0.$$

As mentioned earlier, part (a) of Theorem 1.1 is known to experts in the relativity community; (b) was first proved by Huang [11] (under slightly different decay assumptions). Our contribution is to provide more elementary and simpler proofs.

The structure of the paper is as follows. In the next section, we prove part (a) of Theorem 1.1. In section 3, we prove part (b).

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## 2. $\mathbf{m} = \mathbf{m}_I$

On an asymptotically flat end  $(M^n, g)$ , as our discussion is only near the infinity, we may extend  $(M^n, g)$  so that  $M$  is diffeomorphic to  $\mathbb{R}^n$ . We will assume this throughout the rest of the paper.

**Definition 1.** Let  $f$  be a function defined near infinity of  $\mathbb{R}^n$ . We say that  $f = o_k(|x|^{-\tau})$ , if  $f$  is in  $C^k$  and  $|x|^{|\alpha|+\tau}|\partial^\alpha f(x)| = o(1)$  as  $x \rightarrow \infty$  for all  $\alpha$  with  $0 \leq |\alpha| \leq k$ .

**Theorem 2.1.** Suppose  $g_{ij} - \delta_{ij} = o_2(|x|^{-\frac{n-2}{2}})$  on  $(M^n, g)$ . Then

$$\lim_{r \rightarrow \infty} (\mathbf{m}(r) - \mathbf{m}_I(r)) = 0.$$

In fact, the following is true. Let  $\{D_l\}_{l=1}^\infty$  be a sequence of bounded open sets with Lipschitz boundary  $\Sigma_l := \partial D_l$  which has area  $|\Sigma_l|$ . Let  $r_l := \inf_{x \in \Sigma_l} |x|$ . Assume

$$\lim_{l \rightarrow \infty} r_l = \infty \quad \text{and} \quad |\Sigma_l| \leq C r_l^{n-1}$$

for some constant  $C$  independent of  $l$ . Then

$$\lim_{l \rightarrow \infty} (\mathbf{m}(l) - \mathbf{m}_I(l)) = 0,$$

where  $\mathbf{m}(l)$  is the RHS in (1.1) integrating over  $\Sigma_l$  and  $\mathbf{m}_I(l)$  is the RHS in (1.3) integrating over  $\Sigma_l$ .

*Proof.* The condition  $g_{ij} - \delta_{ij} = o_2(|x|^{-\frac{n-2}{2}})$  shows

$$(2.1) \quad |\nu_e^i - \nu_g^i| = o(r_l^{-\frac{n-2}{2}}), \quad d\sigma_g = \left(1 + o(r_l^{-\frac{n-2}{2}})\right) d\sigma_e$$

on  $\Sigma_l$ , and

(2.2)

$$\begin{aligned} 2R_{ij}(x) &= 2\frac{\partial}{\partial x^k}\Gamma_{ji}^k - 2\frac{\partial}{\partial x^j}\Gamma_{ki}^k + 2\Gamma_{kl}^k\Gamma_{ji}^l - 2\Gamma_{jl}^k\Gamma_{ki}^l \\ &= \frac{\partial}{\partial x^k}(g_{ki,j} + g_{kj,i} - g_{ij,k}) - \frac{\partial}{\partial x^j}(g_{ki,k} + g_{kk,i} - g_{ki,k}) + o(|x|^{-n}) \\ &= g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij} + o(|x|^{-n}) \\ &= o(|x|^{-\frac{n+2}{2}}). \end{aligned}$$

Using the fact that  $|\Sigma_l| \leq Cr_l^{n-1}$ , by (2.2) as  $l \rightarrow \infty$  we have:

$$\begin{aligned}
 (2.3) \quad -2 \int_{\Sigma_l} R_{ij} x^i \nu_g^j d\sigma_g &= -2 \int_{\Sigma_l} R_{ij} x^i \nu_e^j d\sigma_g - 2 \int_{\Sigma_l} R_{ij} x^i (\nu_g^j - \nu_e^j) d\sigma_g \\
 &= -2 \int_{\Sigma_l} R_{ij} x^i \nu_e^j d\sigma_e + o(1) \\
 &= \int_{\Sigma_l} (-g_{ki,kj} - g_{kj,ki} + g_{ij,kk} + g_{kk,ij}) x^i \nu_e^j d\sigma_e + o(1).
 \end{aligned}$$

We claim that

$$\begin{aligned}
 (2.4) \quad &\int_{\Sigma_l} (-g_{ki,kj} - g_{kj,ki} + g_{ij,kk} + g_{kk,ij}) x^i \nu_e^j d\sigma_e \\
 &= (n-2) \int_{\Sigma_l} (g_{kj,k} - g_{kk,j}) \nu_e^j d\sigma_e + \int_{\Sigma_l} (-g_{kj,kj} + g_{kk,jj}) x^i \nu_e^i d\sigma_e.
 \end{aligned}$$

If the claim is true, then by (2.1) and (2.3), we have

$$\begin{aligned}
 (2.5) \quad &-2 \int_{\Sigma_l} R_{ij} x^i \nu_g^j d\sigma_g \\
 &= (n-2) \int_{\Sigma_l} (g_{kj,k} - g_{kk,j}) \nu_g^j d\sigma_g + \int_{\Sigma_l} (-g_{kj,kj} + g_{kk,jj}) x^i \nu_e^i d\sigma_e + o(1).
 \end{aligned}$$

To verify (2.4), viewing each  $g_{ij}$  as functions on  $\mathbb{R}^n$ , we may find sequences of smooth functions  $\{g_{ij}^{(m)}\}$  such that  $g_{ij}^{(m)} = g_{ji}^{(m)}$  for all  $m$  and  $\{g_{ij}^{(m)}\}$  converges to  $g_{ij}$  uniformly in  $C^2$  norm on any compact sets. Hence to prove (2.4), we may assume that  $g$  is  $C^3$ . Integrating by parts

on  $D_l$  (since  $\partial D_l$  is Lipschitz), we have

$$\begin{aligned}
& \int_{\Sigma_l} (-g_{ki,kj} - g_{kj,ki} + g_{ij,kk} + g_{kk,ij}) x^i \nu_e^j d\sigma_e \\
& \int_{D_l} \frac{\partial}{\partial x^j} [(-g_{ki,kj} - g_{kj,ki} + g_{ij,kk} + g_{kk,ij}) x^i] dv_e \\
& = \int_{D_l} (-g_{ki,kj} - g_{kj,ki} + g_{ij,kk} + g_{kk,ij}) \delta_j^i dv_e \\
& \quad + \int_{D_l} (-g_{ki,kjj} - g_{kj,kij} + g_{ij,kkj} + g_{kk,ijj}) x^i dv_e \\
& = 2 \int_{D_l} (g_{kk,jj} - g_{kj,kj}) dv_e + \int_{D_l} (-g_{kj,kij} + g_{kk,ijj}) x^i dv_e \\
& \quad (\text{since } \sum_{k,j} (-g_{ki,kjj} + g_{ij,kkj}) = 0) \\
& = (n-2) \int_{D_l} (g_{kj,kj} - g_{kk,jj}) dv_e + \int_{D_l} \frac{\partial}{\partial x^i} ((-g_{kj,kj} + g_{kk,jj}) x^i) dv_e \\
& = (n-2) \int_{\Sigma_l} (g_{kj,k} - g_{kk,j}) \nu_e^j d\sigma_e + \int_{\Sigma_l} (-g_{kj,kj} + g_{kk,jj}) x^i \nu_e^i d\sigma_e,
\end{aligned}$$

where  $dv_e$  is the volume element with respect to the Euclidean metric. This proves (2.4).

On the other hand, by (2.2), we have

$$\begin{aligned}
(2.6) \quad R_g(x) &= \sum_i R_{ii} + o(|x|^{-n}) \\
&= \frac{1}{2} \sum_{i,k} (g_{ki,ki} + g_{ki,ki} - g_{ii,kk} - g_{kk,ii}) + o(|x|^{-n}) \\
&= g_{ik,ik} - g_{kk,ii} + o(|x|^{-n}) \\
&= o(|x|^{-\frac{n+2}{2}}).
\end{aligned}$$

Hence

$$\begin{aligned}
(2.7) \quad \int_{\Sigma_l} (-g_{kj,kj} + g_{kk,jj}) x^i \nu_e^i d\sigma_e &= - \int_{\Sigma_l} R_g x^i \nu_e^i d\sigma_e + o(1) \\
&= - \int_{\Sigma_l} R_g g(X, \nu_g) d\sigma_g + o(1).
\end{aligned}$$

Combining this with (2.5), we conclude that

$$\begin{aligned}
(2.8) \quad -2 \int_{\Sigma_l} \left( \text{Ric} - \frac{1}{2} R_g g \right) (X, \nu_g) d\sigma_g &= (n-2) \int_{\Sigma_l} (g_{kj,k} - g_{kk,j}) \nu_g^j d\sigma_g + o(1)
\end{aligned}$$

as  $l \rightarrow \infty$ . From this it is easy to see the theorem is true.  $\square$

### 3. $\mathbf{c}_{\text{CS}} = \mathbf{c}_I$

**Definition 2.** For a function  $f(x)$  defined on  $\mathbb{R}^n$ , let

$$f^{\text{odd}}(x) := \frac{1}{2}(f(x) - f(-x)), \quad f^{\text{even}}(x) := \frac{1}{2}(f(x) + f(-x)).$$

**Theorem 3.1.** Suppose  $g_{ij} - \delta_{ij} = o_2(|x|^{-\frac{n-2}{2}})$  on  $(M^n, g)$ . Suppose  $g$  also satisfies the Regge-Teitelboim condition:  $g_{ij}^{\text{odd}}(x) = o_2(|x|^{-\frac{n}{2}})$ . Then

$$\lim_{r \rightarrow \infty} (\mathbf{c}_{\text{CS}}(r) - \mathbf{c}_I(r)) = 0.$$

*Proof.* For  $x \in S_r$ , we have the following:

$$(3.1) \quad \begin{cases} g_{ij}(x) = (g_{ij})^{\text{even}}(x) + o(|x|^{-\frac{n}{2}}); \\ \frac{\partial g_{ij}}{\partial x^k}(x) = \left(\frac{\partial g_{ij}}{\partial x^k}\right)^{\text{odd}}(x) + o(|x|^{-1-\frac{n}{2}}); \\ \Gamma_{ij}^k(x) = (\Gamma_{ij}^k)^{\text{odd}}(x) + o(|x|^{-1-\frac{n}{2}}); \\ R_{ij}(x) = (R_{ij})^{\text{even}}(x) + o(|x|^{-2-\frac{n}{2}}); \\ R_g(x) = (R_g)^{\text{even}}(x) + o(|x|^{-2-\frac{n}{2}}). \end{cases}$$

We also have

$$(3.2) \quad \nu_g^i(x) - \nu_e^i(x) = (\nu_g^i - \nu_e^i)^{\text{odd}}(x) + o(|x|^{-\frac{n}{2}}),$$

$$(3.3) \quad f(x) - 1 = (f - 1)^{\text{even}}(x) + o(|x|^{-\frac{n}{2}}),$$

where  $f(x)$  is defined by  $d\sigma_g(x) = f(x)d\sigma_e(x)$ . For each  $\alpha = 1, \dots, n$ , define  $Y_{(\alpha)}(x) = Y_{(\alpha)}^i \frac{\partial}{\partial x^i}$ , where  $Y_{(\alpha)}^i = (|x|^2 \delta^{\alpha i} - 2x^\alpha x^i)$ . Note that  $Y^i(x) = Y^i(-x)$ . By (3.1)–(3.3), we have as  $r \rightarrow \infty$

$$(3.4) \quad \begin{aligned} \int_{S_r} \text{Ric}(Y, \nu_g) d\sigma_g &= \int_{S_r} \text{Ric}(Y, \nu_g - \nu_e)(f - 1) d\sigma_e + \int_{S_r} \text{Ric}(Y, \nu_g - \nu_e) d\sigma_e \\ &\quad + \int_{S_r} \text{Ric}(Y, \nu_e)(f - 1) d\sigma_e + \int_{S_r} \text{Ric}(Y, \nu_e) d\sigma_e \\ &= \int_{S_r} \text{Ric}(Y, \nu_e) d\sigma_e + o(1), \end{aligned}$$

where we also used the fact  $g_{ij}(x) - \delta_{ij} = o_2(|x|^{-\frac{n-2}{2}})$ . Using (3.1) and (2.2), as  $r \rightarrow \infty$ , we then have

$$\begin{aligned}
(3.5) \quad & 2 \int_{S_r} R_{ij} Y^i \nu_e^j d\sigma_e \\
&= 2 \int_{S_r} \left( \frac{\partial}{\partial x^k} \Gamma_{ji}^k - \frac{\partial}{\partial x^j} \Gamma_{ki}^k \right) Y^i \nu_e^j d\sigma_e + o(1) \\
&= \int_{S_r} \left[ (g^{ks} (g_{is,jk} + g_{js,ik} - g_{ij,sk}) - g^{ks} (g_{ks,ij} + g_{is,kj} - g_{ki,sj})) \right] Y^i \nu_e^j d\sigma_e + o(1) \\
&= \int_{S_r} (g_{ik,jk} + g_{jk,ik} - g_{ij,kk} - g_{kk,ij}) Y^i \nu_e^j d\sigma_e + o(1).
\end{aligned}$$

As in the proof of Theorem 2.1, we may assume that  $g$  is smooth to obtain:

$$\begin{aligned}
& \int_{S_r} (g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) Y^i \nu_e^j d\sigma_e \\
&= \int_{B(r)} \frac{\partial}{\partial x^j} [(g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) Y^i] dv_e \\
&= \int_{B(r)} (g_{kj,ki} - g_{kk,ij}) Y^i dv_e + \int_{B(r)} (g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) \frac{\partial}{\partial x^j} Y^i dv_e \\
&= \int_{S_r} (g_{kj,kj} - g_{kk,jj}) Y^i \nu_e^i d\sigma_e - \int_{B(r)} (g_{kj,kj} - g_{kk,jj}) \frac{\partial}{\partial x^i} Y^i dv_e \\
&\quad + \int_{B(r)} (g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) \frac{\partial}{\partial x^j} Y^i dv_e.
\end{aligned}$$

Since

$$\frac{\partial}{\partial x^j} Y^i = \frac{\partial}{\partial x^j} (|x|^2 \delta^{\alpha i} - 2x^\alpha x^i) = 2x^j \delta^{\alpha i} - 2\delta^{\alpha j} x^i - 2x^\alpha \delta_j^i,$$

we have

$$(g_{kj,kj} - g_{kk,jj}) \frac{\partial}{\partial x^i} Y^i = -2nx^\alpha (g_{kj,kj} - g_{kk,jj})$$

and

$$\begin{aligned}
(g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) \frac{\partial}{\partial x^j} Y^i &= -2x^\alpha (g_{ki,ki} + g_{ki,ki} - g_{ii,kk} - g_{kk,ii}) \\
&= -4x^\alpha (g_{ki,ki} - g_{kk,ii}).
\end{aligned}$$



Hence,

$$\begin{aligned}
(3.6) \quad & \int_{S_r} (g_{ki,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij}) Y^i \nu_e^j d\sigma_e \\
&= \int_{S_r} (g_{kj,kj} - g_{kk,jj}) Y^i \nu_e^j d\sigma_e + 2(n-2) \int_{B(r)} x^\alpha (g_{ki,ki} - g_{kk,ii}) dv_e \\
&= \int_{S_r} (g_{kj,kj} - g_{kk,jj}) Y^i \nu_e^i d\sigma_e + 2(n-2) \int_{S_r} x^\alpha (g_{ki,k} - g_{kk,i}) \nu_e^i d\sigma_e \\
&\quad - 2(n-2) \int_{B(r)} (g_{k\alpha,k} - g_{kk,\alpha}) dv_e \\
&= \int_{S_r} (g_{kj,kj} - g_{kk,jj}) Y^i \nu_e^i d\sigma_e + 2(n-2) \int_{S_r} [x^\alpha (g_{ki,k} - g_{kk,i}) \nu_e^i - (g_{k\alpha} \nu_e^k - g_{kk} \nu_e^\alpha)] d\sigma_e.
\end{aligned}$$

Here  $g_{k\alpha}, g_{kk}$  in the second term in the integrand of the second integral can be replaced by  $h_{k\alpha}, h_{kk}$  where  $h_{ij} := g_{ij} - \delta_{ij}$ .

Using (3.1)–(3.3), we may argue as before to conclude that

$$\int_{S_r} (g_{kj,kj} - g_{kk,jj}) Y^i \nu_e^i d\sigma_e = \int_{S_r} R_g g(Y, \nu_g) d\sigma_g + o(1),$$

and

$$\begin{aligned}
& \int_{S_r} [x^\alpha (g_{ki,k} - g_{kk,i}) \nu_e^i - (g_{k\alpha} \nu_e^k - g_{kk} \nu_e^\alpha)] d\sigma_e \\
&= \int_{S_r} [x^\alpha (g_{ki,k} - g_{kk,i}) \nu_g^i - (g_{k\alpha} \nu_g^k - g_{kk} \nu_g^\alpha)] d\sigma_g + o(1)
\end{aligned}$$

as  $r \rightarrow \infty$ . Combining these with (3.4) and (3.5), and using (2.6) for the expression of  $R_g$ , we have

$$\begin{aligned}
(3.7) \quad & \int_{S_r} \left( R_{ij} - \frac{1}{2} R_g g_{ij} \right) Y^i \nu_g^j d\sigma_g \\
&= (n-2) \int_{S_r} [x^\alpha (g_{ki,k} - g_{kk,i}) \nu_e^i d\sigma_e - (g_{k\alpha} \nu_e^k - g_{kk} \nu_e^\alpha)] d\sigma_e + o(1) \\
&= (n-2) \int_{S_r} [x^\alpha (g_{ki,k} - g_{kk,i}) \nu_g^i - (g_{k\alpha} \nu_g^k - g_{kk} \nu_g^\alpha)] d\sigma_g + o(1)
\end{aligned}$$

as  $r \rightarrow \infty$ . From this the result follows.  $\square$

*Remark 3.1.* On an asymptotically flat  $(M^n, g)$  with  $g_{ij} - \delta_{ij} = O_2(|x|^{-q})$ , where  $q > \frac{n-2}{2}$ , it follows from the proof of Theorem 2.1 in Section 2 that  $\mathbf{m}$  and hence  $\mathbf{m}_I$  are defined if and only if  $\lim_{r \rightarrow \infty} \int_{B(r)} R_g dv_g$  exists.

On the other hand, if in addition  $g^{\text{odd}} = O_2(|x|^{-q-1})$ , the computation in [8] (also cf. [6]) shows  $\mathbf{c}_{\text{CS}}$  and hence  $\mathbf{c}_I$  are defined if and only if  $\lim_{r \rightarrow \infty} \int_{B(r)} x^i R_g dv_g$  exists for  $i = 1, \dots, n$ . Here  $f = O_2(|x|^{-q})$  means  $|x|^{|\alpha|+q} |\partial^\alpha f| \leq C$  for all  $\alpha$  with  $|\alpha| \leq 2$ .

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